

GEOMETRIC STRUCTURES MODELED ON SMOOTH PROJECTIVE HOROSPHERICAL VARIETIES OF PICARD NUMBER ONE

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ABSTRACT. Geometric structures modeled on rational homogeneous manifolds are studied to characterize rational homogeneous manifolds and to prove their deformation rigidity. To generalize these characterizations and deformation rigidity results to quasihomogeneous varieties, we first study horospherical varieties and geometric structures modeled on horospherical varieties. Using Cartan geometry, we prove that a geometric structure modeled on a smooth projective horospherical variety of Picard number one is locally equivalent to the standard geometric structure when the geometric structure is defined on a Fano manifold of Picard number one.

INTRODUCTION

Let M be a Fano manifold of Picard number one. An irreducible component \mathcal{K} of the space of rational curves on M is called a *minimal dominating component* if the subvariety \mathcal{K}_x consisting of members that pass through x is nonempty and projective for a general point $x \in M$. The tangent directions at x of members of \mathcal{K}_x form a subvariety \mathcal{C}_x of $\mathbb{P}T_x(M)$, which is called the *variety of minimal rational tangents at x* . Many techniques can be used to study the projective geometries of $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ which are believed to control the geometry of the manifold M . In this paper, we study geometric structures modeled on horospherical varieties which we expect to get from the variety of minimal rational tangents.

When S is a rational homogeneous manifold of Picard number one, a pair of the automorphism group of the variety of minimal rational tangent \mathcal{C}_s and the linear span D_s of the cone $\hat{\mathcal{C}}_s \subset T_s(S)$ of \mathcal{C}_s for $s \in S$ corresponds to

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the standard geometric structure on S . Jun-Muk Hwang, Ngaiming Mok, and Jaehyun Hong published significant work on the geometric structures modeled on S that arise from the variety of minimal rational tangents. They published work on Hermitian symmetric manifolds and homogeneous contact manifolds in [8], [5], and [16], and on other rational homogeneous manifolds associated with long simple roots in [6].

Theorem 0.1 ([8], [16] and [6]). *Let $S = G/P$ where G is a simple Lie group and P is a maximal parabolic subgroup associated with a long root. Let $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at a base point $s \in S$. Let M be a Fano manifold of Picard number one and \mathcal{C}_x be the variety of minimal rational tangents at a general point $x \in M$ associated with a minimal dominating rational component \mathcal{K} . Suppose that $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ and $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ are isomorphic as projective subvarieties for a general point $x \in M$. Then, M is biholomorphic to S .*

It is natural to ask what happens when we replace rational homogeneous manifolds with quasihomogeneous varieties, especially with smooth projective horospherical varieties of Picard number one. Horospherical varieties are complex normal algebraic varieties where a connected complex reductive algebraic group acts with an open orbit isomorphic to a torus bundle over a flag variety. Boris Pasquier classified smooth projective horospherical varieties of Picard number one in his paper [19]. When a smooth horospherical variety is homogeneous, it is isomorphic to one of quadrics Q^{2m} , Grassmannians $\mathrm{Gr}(i+1, m+2)$, and spinor varieties $\mathrm{Spin}(2m+1)/P_{\alpha_m}$. These are compact irreducible Hermitian symmetric manifolds, and the geometric structures modeled on them have already been studied in Theorem 0.1.

In this paper, we will study geometric structures modeled on smooth nonhomogeneous projective horospherical varieties of Picard number one.

Main Theorem 0.2 (Theorem 4.7). *Let X be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let M be a Fano manifold of Picard number one. Then, any geometric structure of M modeled on X is locally equivalent to the standard geometric structure on X .*

We use Definition 4.3 for the definition of a geometric structure modeled on X . We will prove the existence of Cartan connections (Proposition 4.4) and use it to prove local equivalence of geometric structures modeled on smooth nonhomogeneous projective horospherical varieties of Picard number one.

Noboru Tanaka ([21], [22]) and Tohru Morimoto ([17]) found the sufficient conditions for the existence of Cartan connections associated with geometric structures having certain symmetries, such as geometric structures modeled on rational homogeneous manifolds. We generalize these conditions for some quasihomogeneous manifold cases in Theorem 1.15.

To prove the existence of Cartan connections associated with geometric structures modeled on X , we need to study the Lie algebra $\mathfrak{g} := \mathfrak{aut}(X)$ of the automorphism group of X . In particular, it is important to know whether \mathfrak{g} satisfies the prolongation property. Keizo Yamaguchi showed that \mathfrak{g} satisfies the prolongation property associated with a fundamental graded Lie subalgebra \mathfrak{m} of \mathfrak{g} by proving that the Lie algebra cohomology space $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ vanishes. When X is a rational homogeneous manifold, \mathfrak{g} is a semisimple Lie algebra, and thus we can apply Kostant's harmonic theory to the Lie algebra cohomology spaces. However, in our case, \mathfrak{g} is not semisimple and we cannot apply Kostant's harmonic theory directly.

Colleen Robles and Dennis The ([20]) computed Lie algebra cohomology spaces for some cases in which \mathfrak{g} is not semisimple by modifying Kostant's harmonic theory. In some ways, our embedding of \mathfrak{g} into $\mathfrak{gl}(V)$ is similar to the embedding of a parabolic subalgebra into a simple graded Lie algebra of depth one, which appears in their paper [20]. It would be interesting if one can generalize Kostant's harmonic theory fully to the case where \mathfrak{g} is not semisimple. Instead of generalizing the whole of Kostant's harmonic theory, we reduce the vanishing of Lie algebra cohomology spaces to the vanishing of Lie algebra cohomology spaces associated with a maximal semisimple subalgebra of \mathfrak{g} , which now can be computed using Kostant's harmonic theory.

The remainder of this paper is organized as follows. In Section 1, we review the theory of Cartan connections. In Section 2, we prove that $\mathfrak{aut}(X)$ satisfies the conditions for the existence of Cartan connections except the condition that the first generalized Spencer cohomology vanishes. In Section 3, we compute the vanishing of the first generalized Spencer cohomology spaces of $\mathfrak{aut}(X)$. In Section 4, we prove the local flatness of the geometric structure modeled on X and complete the proof of our main theorem.

We work over the complex number field \mathbb{C} without any additional mention of a number field. All manifolds, Lie groups, and Lie algebras will be understood as complex manifolds, complex Lie groups, and complex Lie algebras, respectively.

1. PROLONGATIONS AND CARTAN CONNECTIONS

1.1. Prolongations.

Definition 1.1. Let \mathfrak{g} be a Lie algebra. A *gradation* of \mathfrak{g} is a direct decomposition $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for $p, q \in \mathbb{Z}$. A *fundamental graded Lie algebra* is a nilpotent graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ generated by \mathfrak{g}_{-1} , i.e., $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Given a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$, there exists a unique graded Lie algebra $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ such that

- (1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$
- (2) if $z \in \mathfrak{g}_p(\mathfrak{m})$ for $p \geq 0$ satisfies $[z, \mathfrak{m}] = 0$, then $z = 0$.
- (3) $\mathfrak{g}(\mathfrak{m})$ is the largest graded Lie algebra satisfying conditions (1) and (2).

We refer to $\mathfrak{g}(\mathfrak{m})$ as the *universal prolongation* of \mathfrak{m} . Let $\mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{m})$ be a subalgebra. Then, the *prolongation* of $(\mathfrak{m}, \mathfrak{g}_0)$ is the largest graded Lie subalgebra $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_p \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0)$ of $\mathfrak{g}(\mathfrak{m})$ such that $\bigoplus_{p < 0} \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m}$ and $\mathfrak{g}_0(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{g}_0$.

Definition 1.2. Let \mathfrak{m} be a Lie algebra and Γ be a vector space. Let $\gamma: \mathfrak{m} \rightarrow \text{End}(\Gamma)$ be a representation of \mathfrak{m} .

Define the *coboundary operator* $\partial: \text{Hom}(\wedge^q \mathfrak{m}, \Gamma) \rightarrow \text{Hom}(\wedge^{q+1} \mathfrak{m}, \Gamma)$ as follows: for $\phi \in \text{Hom}(\wedge^q \mathfrak{m}, \Gamma)$,

$$\begin{aligned} \partial\phi(z_1 \wedge \cdots \wedge z_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} \gamma(z_i) \phi(z_1 \wedge \cdots \wedge \hat{z}_i \cdots \wedge z_{q+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \phi([z_i, z_j] \wedge z_1 \wedge \cdots \wedge \hat{z}_i \cdots \wedge \hat{z}_j \cdots \wedge z_{q+1}), \end{aligned}$$

where \hat{z}_i denotes skipping z_i . We denote the induced cochain complex by $(C(\mathfrak{m}, \Gamma), \partial)$ and the derived space of the cohomology by $H(\mathfrak{m}, \Gamma)$.

Definition 1.3. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra. Let $\Gamma = \bigoplus_{p \in \mathbb{Z}} \Gamma_p$ be a graded vector space and let $\gamma: \mathfrak{m} \rightarrow \text{End}(\Gamma)$ be a representation of \mathfrak{m} on Γ such that $\gamma(\mathfrak{g}_p)\Gamma_q \subset \Gamma_{p+q}$ for $p < 0$ and $q \in \mathbb{Z}$. The cochain complex $(C(\mathfrak{m}, \Gamma), \partial)$ has the following bigradation (Section 1

of [22] and Section 2.4 of [24]):

$$\begin{aligned} C^{p,0}(\mathfrak{m}, \Gamma) &= \bigoplus_p \Gamma_{p-1} \\ C^{p,q}(\mathfrak{m}, \Gamma) &= \bigoplus_{j \leq -q} \text{Hom}(\wedge_j^q \mathfrak{m}, \Gamma_{j+p+q-1}), \end{aligned}$$

where

$$\wedge_j^q \mathfrak{m} = \bigoplus_{\substack{i_1 + \dots + i_q = j \\ i_k < 0}} \mathfrak{g}_{i_1} \wedge \dots \wedge \mathfrak{g}_{i_q}.$$

Then the the coboundary operator ∂ maps $C^{p+1,q}(\mathfrak{m}, \Gamma)$ to $C^{p,q+1}(\mathfrak{m}, \Gamma)$ and the cohomology space with the bigradation

$$H^q(\mathfrak{m}, \Gamma) = \bigoplus_p H^{p,q}(\mathfrak{m}, \Gamma)$$

is called *the generalized Spencer cohomology space* of (Γ, \mathfrak{m}) .

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a graded Lie subalgebra of \mathfrak{g} , which is fundamental. Let $\text{ad}: \mathfrak{m} \rightarrow \text{End}(\mathfrak{g})$ be the adjoint representation of \mathfrak{m} on \mathfrak{g} such that, for $z_1 \in \mathfrak{m}$ and $z_2 \in \mathfrak{g}$, $\text{ad}(z_1)z_2 = [z_1, z_2]$. The following is an effective way to show that a given graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of \mathfrak{m} (or of $(\mathfrak{m}, \mathfrak{g}_0)$) related with the first (generalized) Spencer cohomology $H^1(\mathfrak{m}, \mathfrak{g}) = \bigoplus_p H^{p,1}(\mathfrak{m}, \mathfrak{g})$.

Lemma 1.4 (Lemma 2.1 of [24]). *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is fundamental. Then, \mathfrak{g} is the prolongation of \mathfrak{m} (resp. of $(\mathfrak{m}, \mathfrak{g}_0)$) if and only if the following two conditions hold:*

- (1) *if $z \in \mathfrak{g}_p$ for $p \geq 0$, satisfies $[z, \mathfrak{m}] = 0$, then $z = 0$.*
- (2) *$H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p \geq 0$ (resp. $p \geq 1$).*

1.2. Cartan connections.

Definition 1.5. Let M be a manifold and TM be the tangent bundle of M . A *tangential filtration* $F = \{F^p\}_{p \in \mathbb{Z}_{\leq 0}}$ on M is a sequence of subbundles $F^p = F^p TM$ of TM satisfying the following:

- (1) $F^{p+1} \subset F^p$
- (2) $F^0 = 0$ and $\cup F^p = TM$
- (3) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for $p, q \in \mathbb{Z}_{\leq 0}$,

where \mathcal{F}^\bullet is the sheaf of sections of F^\bullet . We refer to (M, F) as a *filtered manifold*.

The *symbol algebra* $\text{Symb}_x(F) = \bigoplus_{p \in \mathbb{Z}_{\leq 0}} \text{Symb}_x^p(F)$ of F at $x \in M$ is given by

$$\text{Symb}_x^p(F) = F_x^p TM / F_x^{p+1} TM$$

with a natural bracket induced by the Lie bracket of vector fields on M .

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra. A filtered manifold (M, F) is called a *regular filtered manifold of type \mathfrak{m}* if the symbol algebras $\text{Symb}_x(F)$ are all isomorphic to \mathfrak{m} for all $x \in M$.

Definition 1.6. Let (M, F) be a regular filtered manifold of type \mathfrak{m} . Let $\mathcal{R}_x(M, \mathfrak{m})$ be the set of all isomorphisms $\varsigma : \mathfrak{m} \rightarrow \text{Symb}_x(F)$ of graded Lie algebras. Then, with the structure group $G_0(\mathfrak{m})$ which consists of all automorphisms of the graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$, $\mathcal{R} := \bigcup_{x \in M} \mathcal{R}_x(M, \mathfrak{m})$ is a principal $G_0(\mathfrak{m})$ -bundle on M .

$$\begin{array}{c} G_0(\mathfrak{m}) \rightarrow \mathcal{R} = \bigcup_{x \in M} \mathcal{R}_x(M, \mathfrak{m}) \\ \downarrow \\ M \end{array}$$

This fiber bundle \mathcal{R} is called the *frame bundle of the regular filtered manifold (M, F) of type \mathfrak{m}* , or simply the *frame bundle of (M, F)* .

Given a closed subgroup $G_0 \subset G_0(\mathfrak{m})$, a G_0 -*structure* on (M, F) is a G_0 -subbundle of the frame bundle \mathcal{R} . Two G_0 -structures on (M_1, F_1) and (M_2, F_2) are *locally equivalent* if there exist two open subsets U_1 of M_1 and U_2 of M_2 , and a G_0 -bundle isomorphism over the open subsets U_1 and U_2 .

Definition 1.7. A *differential system* (M, D) on manifold M is a subbundle D of the tangent bundle $T(M)$ of M . The subbundle D is *completely integrable* if and only if $[D, D] \subset D$. For a non-integrable differential system D , we consider the *derived system* ∂D of D which is defined in terms of sections by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}],$$

where \mathcal{D} denotes the space of sections of D . Moreover, the k -th weak derived systems $\partial^{(k)} D$ of D are inductively defined by

$$\partial^{(k)} \mathcal{D} = \partial^{(k-1)} \mathcal{D} + [\mathcal{D}, \partial^{(k-1)} \mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)} \mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$. A differential system (M, D) is said to be *regular* if $D^{-(k+1)} := \partial^{(k)} D$ is a

subbundle of $T(M)$ for every integer $k \geq 1$. For a regular differential system (M, D) such that $D^{-\mu} = T(M)$, we define the associated graded algebra $\mathfrak{m}(x)$ at $x \in M$, which was introduced by Noboru Tanaka in [21]. We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ (for $p < -1$) and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then, $\mathfrak{m}(x)$ becomes a fundamental graded Lie algebra, which we refer to as *the symbol algebra* of (M, D) at $x \in M$. If the symbol algebra $\mathfrak{m}(x)$ is isomorphic to a given fundamental graded Lie algebra for each $x \in M$, then we refer to (M, D) as a *regular differential system of type \mathfrak{m}* .

Definition 1.8. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra. A regular tangential filtration (M, F) of type \mathfrak{m} derived from a regular differential system (M, D) of type \mathfrak{m} is $F^p = D^p$ for $p \leq 0$. We just denote (M, D) as a *regular filtered manifold derived from a regular differential system* (M, D) of type \mathfrak{m} . A G_0 -structure on (M, D) is a G_0 -subbundle of the frame bundle \mathcal{R} of (M, D) .

Definition 1.9. Let \mathfrak{g} be a Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Let H be a connected Lie group with Lie algebra \mathfrak{h} and let $\text{Ad}: H \rightarrow GL(\mathfrak{g})$ be the adjoint representation of H on \mathfrak{g} . A *Cartan connection of type (\mathfrak{g}, H)* on a manifold M is a principal H -bundle $\pi: P \rightarrow M$ with \mathfrak{g} -valued 1-form ω on P such that

- (1) $\omega(z^\dagger) = z$ for $z \in \mathfrak{h}$ where z^\dagger denotes the fundamental vector field on P induced by $z \in \mathfrak{h}$;
- (2) $R_h^* \omega = \text{Ad}(h^{-1})\omega$ for $h \in H$ where $R_h: P \rightarrow P$ is the right action of $h \in H$ on P ;
- (3) the linear map $\omega_p: T_p(P) \rightarrow \mathfrak{g}$ is a vector space isomorphism for each $p \in P$.

Two Cartan connections of type (\mathfrak{g}, H) , denoted by pairs (P_1, ω_1) on M_1 and (P_2, ω_2) on M_2 , are *locally equivalent* if there exist two open subsets U_1 of M_1 and U_2 of M_2 , and a biholomorphic map $\phi: P_1|_{U_1} \rightarrow P_2|_{U_2}$ descending to $U_1 \rightarrow U_2$ such that $\phi^* \omega_2 = \omega_1$. A Cartan connection of type (\mathfrak{g}, H) is *locally flat* if it is locally equivalent to the Cartan connection on the principal H -bundle $G \rightarrow G/H$ with the Maurer-Cartan form on G where G is a connected Lie group with Lie algebra \mathfrak{g} and an inclusion $H \subset G$ as a closed subgroup.

Let V and W be vector spaces with filtration. Then

$$\begin{aligned} F^p \operatorname{Hom}(V, W) &= \{ \alpha \in \operatorname{Hom}(V, W) \mid \alpha(F^i V) \subset F^{i+p} W \ \forall i \} \\ F^p GL(V) &= \{ \alpha \in GL(V) \mid \alpha - 1_V \in F^p \operatorname{Hom}(V, V) \} \\ F^p \operatorname{Aut}(V) &= \operatorname{Aut}(V) \cap F^p GL(V) \end{aligned}$$

Definition 1.10. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra. Let H be a Lie group and $E = \mathfrak{m} \oplus \mathfrak{h}$ where \mathfrak{h} is the Lie algebra of H . An *skeleton* on \mathfrak{m} is a pair (E, H) with a representation ρ of H on E satisfying the followings:

- (1) $\rho(h)z = \operatorname{Ad}(h)z$ for $h \in H$, $z \in \mathfrak{h}$.
- (2) $\rho(h)F^p \mathfrak{m} \subset F^p \mathfrak{m} \oplus \mathfrak{h}$ for $h \in H$ and $p < 0$, where $F^p \mathfrak{m} = \bigoplus_{p \leq i \leq -1} \mathfrak{g}_i$
- (3) There exist an filtration $\{F^p H\}$ on H , and hence there is induced filtration $F^p E = F^p \mathfrak{m} \oplus F^p \mathfrak{h}$ on E , where $F^p \mathfrak{h}$ is the Lie algebra of $F^p H$ satisfying: for $p \leq 0$, $F^p H = H$ and, for $p \geq 0$, the sequences

$$1 \longrightarrow F^{p+1} H \longrightarrow H \xrightarrow{\rho^p} F^0 \operatorname{Aut}(E^{(p-1)}) / F^{p+1} \operatorname{Aut}(E^{(p-1)})$$

are exact, where $E^{(p-1)} = E / F^p E$, and ρ^p is the homomorphism induced by ρ .

Definition 1.11. Let (M, F) be a regular filtered manifold of type \mathfrak{m} and (E, H) be a skeleton on \mathfrak{m} . A *tower* P on M with skeleton (E, H) is a principal H -bundle $\pi : P \rightarrow M$ with an E -valued 1 form θ satisfying:

- (1) the linear map $\theta_p : T_p(P) \rightarrow E$ is a filtered vector space isomorphism for each $p \in P$;
- (2) $\theta(z^\dagger) = z$ for $z \in \mathfrak{h}$ where z^\dagger denotes the fundamental vector field on P induced by $z \in \mathfrak{h}$;
- (3) $R_h^* \theta = \operatorname{Ad}(h^{-1}) \theta$ for $h \in H$ where $R_h : P \rightarrow P$ is the right action of $h \in H$ on P .

Let \mathfrak{m} be a fundamental graded Lie algebra. Let G_0 be a connected Lie subgroup of $G_0(\mathfrak{m})$ and let \mathfrak{g}_0 be the Lie algebra corresponding to G_0 . Let $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ be the prolongation of the graded Lie algebra of $(\mathfrak{m}, \mathfrak{g}_0)$. Let \mathfrak{M} be a Lie group having \mathfrak{m} as its Lie algebra. The trivial subbundle $\mathfrak{M} \times G_0$ of the frame bundle $\mathfrak{M} \times G_0(\mathfrak{m})$ is the standard G_0 -structure on $(\mathfrak{M}, \mathfrak{m})$.

By Theorem 2.3.2 and Theorem 3.6.1 of [16], we can construct a tower P on \mathfrak{M} with skeleton $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H := H(\mathfrak{m}, G_0))$. More precisely, by Theorem 2.3.2 of [17], for a given $P^{(0)} := \mathfrak{M} \times G_0$, we can construct a unique tower $\mathcal{R}P^{(0)}$ satisfying the universal property: $\mathcal{R}P^{(0)} / F^1 \mathcal{R}P^{(0)} = P^{(0)}$

and any tower Q on \mathfrak{M} with $Q/F^1Q \subset P^{(0)}$ is embedded in the tower $\mathcal{R}P^{(0)}$. The tower $\mathcal{R}P^{(0)}$ is called *universal tower prolonging* $P^{(0)}$. In the proof of Theorem 3.6.1 of [17], we get a tower $P^{(1)}$ and a surjective map $P^{(1)} \rightarrow \mathcal{R}P^{(0)}/F^2$ where $F^2 = F^2\mathcal{R}P^{(0)}$. Apply these two theorem again to obtain universal tower $\mathcal{R}P^{(1)}$ prolonging $P^{(1)}$, and a tower $P^{(2)}$ with a surjective map $P^{(2)} \rightarrow \mathcal{R}P^{(1)}/F^3$, where $F^3 = F^3\mathcal{R}P^{(1)}$. In this way, starting from $P^{(0)} := \mathfrak{M} \times G_0$, we get a tower P which is the limit of the sequence of the bundles $(P^{(0)}, P^{(1)}, P^{(2)}, \dots)$.

$$\begin{array}{ccccc}
 & & P^{(2)} & & P^{(1)} & & P^{(0)} \\
 & \nearrow & & \searrow & \nearrow & \searrow & \nearrow \\
 \dots & & \mathcal{R}P^{(2)}/F^4 & & \mathcal{R}P^{(1)}/F^3 & & \mathcal{R}P^{(0)}/F^2 \\
 & \searrow & & \nearrow & \searrow & \nearrow & \searrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

are exact. We denote the structure group of P by

$$(1.12) \quad H(\mathfrak{m}, G_0).$$

Let $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$ be the Lie algebra of the structure group $H(\mathfrak{m}, G_0)$ of P . By the construction of P , we see $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{p \geq 0} \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0)$, so that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$. The group G_0 is embedded in $H(\mathfrak{m}, G_0)$ as a closed subgroup.

We define a subspace of $\text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ as

$$F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)) := \{\alpha | \alpha(\mathfrak{g}_i \wedge \mathfrak{g}_j) \subset \bigoplus_{p \geq i+j+1} \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0) \text{ for } i, j < 0\}.$$

There are general conditions for the existence of Cartan connections. Let (M, F) be a regular filtered manifold of type \mathfrak{m} , and let $R^{(0)}$ be a G_0 -structure on (M, F) . Under the assumption of following Theorem, we can construct a principal $H(\mathfrak{m}, G_0)$ -bundle $R \rightarrow M$, which is obtained by extending the first order frame bundle $R^{(0)}$. The principal $H(\mathfrak{m}, G_0)$ -bundle $R \rightarrow M$ is a tower with skeleton $(\mathfrak{g}(\mathfrak{g}_0, \mathfrak{m}), H(\mathfrak{m}, G_0))$ with $\mathfrak{g}(\mathfrak{g}_0, \mathfrak{m})$ -valued 1-form ω . Then (P, ω) is a Cartan connection of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H(\mathfrak{m}, G_0))$. For further details, see Chapters 2 and 3 of [17] and Theorem 2.7 of [22].

Theorem 1.13 (Definition 3.10.1 and Theorem 3.10.1 of [17], and Theorem 2.7 of [22]). *Let (M, F) be a regular filtered manifold of type \mathfrak{m} , and let G_0 be a Lie subgroup of $G_0(\mathfrak{m})$ with Lie algebra \mathfrak{g}_0 . Suppose that there exists a subspace W of $F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ such that*

- (1) $F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)) = W \oplus \partial F^1 \text{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)),$
- (2) W is stable under the action of $H := H(\mathfrak{m}, G_0).$

Then, for each G_0 -structure on (M, F) , we can construct a principal H -bundle $P \rightarrow M$ associated with the G_0 -structure on (M, F) and obtain a Cartan connection (P, ω) of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$. Two G_0 -structures on (M, F) are (locally) equivalent when the associated Cartan connections of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$ are (locally) equivalent.

Theorem 1.14 (Proposition 3.10.1 of [17]). *Let \mathfrak{m} be a fundamental graded Lie algebra. Let G_0 be a Lie subgroup of $G_0(\mathfrak{m})$. Let \mathfrak{g}_0 be the subalgebra of $\mathfrak{g}_0(\mathfrak{m})$ corresponding to G_0 , $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$, and $\mathfrak{h} = \bigoplus_{p \geq 0} \mathfrak{g}_p$ be its non-negative part. Assume that the prolongation \mathfrak{g} is finite-dimensional and that there exist a positive definite bilinear form*

$$(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R},$$

a mapping $\tau : \mathfrak{h} \rightarrow \mathfrak{g}$ and a mapping $\tau_0 : G_0 \rightarrow G_0$ such that

- (1) $(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ for $p \neq q$
- (2) $\tau(\mathfrak{g}_p) \subset \mathfrak{g}_{-p}$ for $p \geq 0$, and $([A, z_1], z_2) = (z_1, [\tau(A), z_2])$ for all $z_1, z_2 \in \mathfrak{g}$ and $A \in \mathfrak{h}$
- (3) $(az_1, z_2) = (z_1, \tau_0(a)z_2)$ for $z_1, z_2 \in \mathfrak{g}$ and $a \in G_0$

Then, there exists a full functor from the category of G_0 -structures of type \mathfrak{m} to the category of Cartan connections of type (\mathfrak{g}, H) , where H is the Lie group $H(\mathfrak{m}, G_0)$ with its Lie algebra \mathfrak{h} .

The following theorem is essentially from Proposition 3.10.1 of [17] and will be applied to the Lie algebras \mathfrak{g} of the automorphism groups of nonhomogeneous smooth horospherical varieties of Picard number one.

Theorem 1.15. *Let \mathfrak{m} be a fundamental graded Lie algebra. Let G_0 be a connected Lie subgroup of $G_0(\mathfrak{m})$ and let \mathfrak{g}_0 be the Lie algebra of G_0 . Let $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ and $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{p \geq 0} \mathfrak{g}_p$ be its non-negative part. Let $H := H(\mathfrak{m}, G_0)$ be a Lie group given as (1.12) with its Lie algebra $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$.*

Assume that the prolongation $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is finite-dimensional. We also assume that there exist a graded Lie algebra $\tilde{\mathfrak{g}}$ that contains $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ as a Lie subalgebra, an $\text{ad}(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ -invariant symmetric bilinear form (\cdot, \cdot) on $\tilde{\mathfrak{g}}$, and a map $\tau : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ satisfying

- (1) $\{\cdot, \cdot\} := -(\cdot, \tau \cdot)$ is a positive definite Hermitian inner product on $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ and $\{\mathfrak{g}_p, \mathfrak{g}_q\} = 0$ if $p \neq q$;

- (2) $\mathfrak{g}_p \subset \tilde{\mathfrak{g}}_p$ for any integer p and $\tau(\mathfrak{g}_p) \subset \tilde{\mathfrak{g}}_{-p}$ for any integer $p \geq 0$;
- (3) $\{[A, z_1], z_2\} = -\{z_1, [\tau(A), z_2]\}$ for $A \in \mathfrak{g}$, $z_1, z_2 \in \tilde{\mathfrak{g}}$;
- (4) there exists $\tau_0: G_0 \rightarrow \tilde{G}_0$ such that and $\{az_1, z_2\} = -\{z_1, \tau_0(a)z_2\}$ for $z_1, z_2 \in \tilde{\mathfrak{g}}$ and $a \in G_0$, where \tilde{G}_0 is a closed subgroup of $G_0(\tilde{\mathfrak{m}}) := \text{Aut}(\bigoplus_{p < 0} \tilde{\mathfrak{g}}_p)$ with its Lie algebra $\tilde{\mathfrak{g}}_0$.

Then, for each G_0 -structure on (M, F) of type \mathfrak{m} , we can construct a Cartan connection (P, ω) of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$ so that two G_0 -structures on (M, F) are (locally) equivalent when the associated Cartan connections of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$ are (locally) equivalent.

Proof. We simplify $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$, $\mathfrak{h} = \mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$. By Theorem 1.13, it is sufficient to show that there exists a subspace W of $F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$ such that

- (1) $F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}) = W \oplus \partial F^1 \text{Hom}(\mathfrak{m}, \mathfrak{g})$,
- (2) W is stable under the action of H .

Let \mathfrak{m}' be the dual of \mathfrak{m} . We extend the bilinear form (\cdot, \cdot) to $\wedge^\bullet \tilde{\mathfrak{g}}$. We identify $\wedge^\bullet \mathfrak{m}'$ and $\wedge^\bullet \tau(\mathfrak{m})$ by defining a map $\eta: \wedge^\bullet \mathfrak{m}' \rightarrow \wedge^\bullet \tau(\mathfrak{m}) \subset \wedge^\bullet \tilde{\mathfrak{g}}$ as the inverse of the isomorphism

$$\begin{aligned} \eta': \wedge^\bullet \tau(\mathfrak{m}) &\rightarrow \wedge^\bullet \mathfrak{m}' \\ a &\mapsto \eta'(a) : z \mapsto (a, z) \text{ for } z \in \wedge^\bullet \mathfrak{m}. \end{aligned}$$

Then, $(\eta(f), z) = f(z)$ for $f \in \wedge^\bullet \mathfrak{m}'$ and $z \in \wedge^\bullet \mathfrak{m}$.

We also extend the bilinear form (\cdot, \cdot) to $\text{Hom}(\wedge^\bullet \tilde{\mathfrak{m}}, \tilde{\mathfrak{g}})$ and define $\{., .\} := -(\cdot, \tau(\cdot))$. By assumption (1), the extended Hermitian inner product $\{., .\}$ is positive definite on $\text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g}) \cong \wedge^\bullet \tau(\mathfrak{m}) \otimes \mathfrak{g}$. Let ∂^* be the formal adjoint of ∂ with respect to the extended Hermitian inner product $\{., .\}$ on $\text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g})$.

Then, we have the direct sum decomposition

$$\text{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g}) = \partial \text{Hom}(\wedge^{q-1} \mathfrak{m}, \mathfrak{g}) \oplus \text{Ker } \partial^*.$$

Let us show that $\text{Ker } \partial^*$ is an invariant subspace for the action of H . Let ρ be the representation of $H \subset G$ on $\text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g})$ and ρ_* be the corresponding adjoint representation of $\mathfrak{h} \subset \mathfrak{g}$ on $\text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g})$. Since any element $a \in H$ is written as

$$a = a_0 \cdot \exp(A)$$

with $a_0 \in G_0$, $A \in F^1\mathfrak{h} := \bigoplus_{p>0} \mathfrak{g}_p$, it suffices to show that

- (a) $\partial^* \circ \rho(a_0) = \rho(a_0) \circ \partial^*$ for $a_0 \in G_0$
- (b) $\partial^* \circ \rho_*(A) = \rho_*(A) \circ \partial^*$ for $A \in F^1\mathfrak{h}$.

Let $\tilde{\partial}$ be the coboundary operator on $\text{Hom}(\wedge^\bullet \tilde{\mathfrak{m}}, \tilde{\mathfrak{g}})$. Let $\tilde{\rho}$ be the representation of \tilde{G}_0 on $\text{Hom}(\wedge^\bullet \tilde{\mathfrak{m}}, \tilde{\mathfrak{g}})$ and $\tilde{\lambda}$ be the adjoint representation of $\tilde{\mathfrak{m}} = \bigoplus_{p<0} \tilde{\mathfrak{g}}_p$ on $\text{Hom}(\wedge^\bullet \tilde{\mathfrak{m}}, \tilde{\mathfrak{g}})$.

In general, we have

$$\tilde{\partial} \circ \tilde{\rho}(b_0) = \tilde{\rho}(b_0) \circ \tilde{\partial} \text{ for } b_0 \in \tilde{G}_0,$$

and

$$(1.16) \quad \tilde{\partial} \circ \tilde{\lambda}(B) = \tilde{\lambda}(B) \circ \tilde{\partial} \text{ for } B \in \tilde{\mathfrak{m}}.$$

We denote $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp = \{z \in \tilde{\mathfrak{g}} \mid \{z, y\} = 0 \text{ for all } y \in \mathfrak{g}\}$. Let λ be the representation of $\tau(F^1\mathfrak{h})$ on $\text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g}) \cong \wedge^\bullet \tau(\mathfrak{m}) \otimes \mathfrak{g}$, which is defined by a composition of the adjoint representation of $\tau(F^1\mathfrak{h})$ on $\wedge^\bullet \tau(\mathfrak{m}) \otimes \tilde{\mathfrak{g}}$ and the projection $\pi: \mathfrak{g} \oplus \mathfrak{g}^\perp \rightarrow \mathfrak{g}$:

$$\lambda = (id \otimes \pi) \circ (\text{ad} \otimes \text{ad}): \tau(F^1\mathfrak{h}) \rightarrow \text{End}(\wedge^\bullet \tau(\mathfrak{m}) \otimes \mathfrak{g}).$$

It follows that for $A \in \mathfrak{h}$ and $\phi, \psi \in \text{Hom}(\wedge^\bullet \mathfrak{m}, \mathfrak{g})$,

$$\begin{aligned} \{\phi, \tilde{\partial} \tilde{\lambda}(\tau A) \psi\} &= \{\phi, \partial \tilde{\lambda}(\tau A) \psi\} \\ &= \{\partial^* \phi, \tilde{\lambda}(\tau A) \psi\} \\ &= \{\partial^* \phi, \lambda(\tau A) \psi\} \\ &= \{\phi, \partial \lambda(\tau A) \psi\} \end{aligned}$$

and hence,

$$\begin{aligned} \{\phi, \partial \lambda(\tau A) \psi\} &= \{\phi, \tilde{\partial} \tilde{\lambda}(\tau A) \psi\} \\ &= \{\phi, \tilde{\lambda}(\tau A) \tilde{\partial} \psi\} \text{ because (1.16) and } \tau(F^1\mathfrak{h}) \subset \tilde{\mathfrak{m}} \text{ by assumption (2)} \\ &= \{\phi, \tilde{\lambda}(\tau A) \partial \psi\} \\ &= \{\phi, \lambda(\tau A) \partial \psi\}. \end{aligned}$$

Thus,

$$(1.17) \quad \partial \circ \lambda(B) = \lambda(B) \circ \partial \text{ for } B \in \tau(F^1\mathfrak{h}).$$

Hence,

$$\begin{aligned}
 \{\partial^* \circ \rho_*(A)\phi, \psi\} &= \{\rho_*(A)\phi, \partial\psi\} \\
 &= -\{\phi, \lambda(\tau A)\partial\psi\} \\
 &= -\{\phi, \partial\lambda(\tau A)\psi\} \text{ from (1.17)} \\
 &= -\{\partial^*\phi, \lambda(\tau A)\psi\} \\
 &= \{\rho_*(A)\partial^*\phi, \psi\},
 \end{aligned}$$

which gives (b).

Similarly, we can verify (a).

If we set $W = \text{Ker } \partial^* \cap F^1 \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$, the proof is completed by Theorem 1.13. \square

2. LIE ALGEBRAS OF THE AUTOMORPHISM GROUPS OF HOROSPHERICAL VARIETIES

Horospherical varieties are complex normal algebraic varieties where a connected reductive algebraic group L acts with an open orbit isomorphic to a torus bundle over a flag variety ([19]).

Theorem 2.1 (Theorem 0.1 and Theorem 1.11 of [19]). *Let X be a smooth nonhomogeneous projective horospherical L -variety with Picard number one. Then, the automorphism group of X is a connected non-reductive linear algebraic group G , acting with exactly two orbits. Moreover, X is uniquely determined by its two closed L -orbits Y and Z , which are isomorphic to L/P_α and L/P_β , respectively. Let π_i be the i -th fundamental weight of L -representation space. The variety $X = (L, \alpha, \beta)$ is one of the triples, with the group G , of the following list.*

- (1) $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$ and $(\text{SO}(2m+1) \times \mathbb{C}^*) \ltimes V(\pi_m)$
- (2) $(B_3, \alpha_1, \alpha_3)$ and $(\text{SO}(7) \times \mathbb{C}^*) \ltimes V(\pi_3)$
- (3) $(C_m, \alpha_i, \alpha_{i+1})$ for $m \geq 2$, $i \in \{1, \dots, m-1\}$ and $((\text{Sp}(2m) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes V(\pi_1)$
- (4) $(F_4, \alpha_2, \alpha_3)$ where α_2 is a long root and $(F_4 \times \mathbb{C}^*) \ltimes V(\pi_4)$
- (5) $(G_2, \alpha_2, \alpha_1)$ and $(G_2 \times \mathbb{C}^*) \ltimes V(\pi_1)$

Here, P_{α_i} is the maximal parabolic subgroup of L associated with the simple root α_i , and $V(\pi_i)$ is the irreducible L -representation with the highest weight π_i .

For a given $X = (L, \alpha, \beta)$, there are irreducible L -representations $V(\pi_\alpha)$ and $V(\pi_\beta)$, and the highest weight vectors v_α of $V(\pi_\alpha)$ and v_β of $V(\pi_\beta)$ such that X is the orbit closure of $L \cdot [v_\alpha + v_\beta] \subset \mathbb{P}(V(\pi_\alpha) + V(\pi_\beta))$ (Section 1.3 of [19]). Hence, X has three orbits under the action of L : one open orbit isomorphic to a torus bundle over $L/(P_\alpha \cap P_\beta)$, and two closed orbits Y and Z that are isomorphic to L/P_α and L/P_β , respectively.

Let G be the automorphism group of X . According to Lemma 1.15 of [19], the closed L -orbit Z is stable under the G -action. Let \tilde{X} be the blowup of X along Z . Then, $G = \text{Aut } \tilde{X}$. According to the proof of Lemma 1.17 of [19], \tilde{X} is a projective bundle over the L -orbit Y and $U \subset G$ acts on \tilde{X} by translation on the fibers of $\tilde{X} \rightarrow Y \cong L/P_\alpha$. Further, $G = (L \times \mathbb{C}^*)/C \ltimes U$, where U is an L -representation space and C is the centralizer.

Proposition 2.2. *Let $X = (L, \alpha, \beta)$ be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let \mathfrak{g} be the Lie algebra of the automorphism group of X . Then,*

- (1) *the Lie algebra \mathfrak{g} is a semidirect product of $(\mathfrak{l} + \mathbb{C})$ and an irreducible \mathfrak{l} -representation U , where \mathfrak{l} is a semisimple Lie algebra, i.e., $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$;*
- (2) *there exist two irreducible L -representations V_α and V_β such that $\mathfrak{l} \subset \text{End}(V_\alpha)$, $\mathfrak{l} \subset \text{End}(V_\beta)$, $\mathbb{C} \simeq \mathbb{C}I \subset \text{End}(V_\beta)$, and $U \subset \text{End}(V_\alpha, V_\beta)$. Hence, we regard \mathfrak{g} as a Lie subalgebra via the inclusion $i: \mathfrak{g} \hookrightarrow \mathfrak{gl}(V) = \text{End } V$ where $V = V_\alpha \oplus V_\beta$. In particular, we can write an element \mathbf{Z} of \mathfrak{g} as*

$$\mathbf{Z} = \begin{pmatrix} l & 0 \\ u & l + c \end{pmatrix} \in \text{End}(V) = \mathfrak{gl}(V)$$

where $l \in \mathfrak{l}$, $u \in U$, and $c \in \mathbb{C}I$.

Let $$ be the operator on $\mathfrak{gl}(V)$ given by $z^* = \bar{z}^t$ for $z \in \mathfrak{gl}(V)$. Let τ be an operator defined by $\tau(z) = -z^*$ for $z \in \mathfrak{gl}(V)$. Let (\cdot, \cdot) be the Cartan-Killing form on $\mathfrak{gl}(V)$.*

- (3) *We define an inner product $\{\cdot, \cdot\}$ by $\{z_1, z_2\} = (z_1, z_2^*) = -(z_1, \tau(z_2))$ for $z_1, z_2 \in \mathfrak{gl}(V)$. Then, a restricted inner product $\{\cdot, \cdot\}$ is a positive definite Hermitian inner product on \mathfrak{g} .*

Proof. (1) It is from Theorem 1.11 of [19].

- (2) It is from the proof of Theorem 1.1. of [19]. Since X is the orbit closure of $L.[v_\alpha + v_\beta] \subset \mathbb{P}(V(\pi_\alpha) + V(\pi_\beta))$, let $V_\alpha = V(\pi_\alpha)$ and $V_\beta = V(\pi_\beta)$.
- (3) If we take two elements \mathbf{Z}_1 and \mathbf{Z}_2 in \mathfrak{g} ,

$$\mathbf{Z}_1 = \begin{pmatrix} l_1 & 0 \\ u_1 & l_1 + c_1 \end{pmatrix} \text{ and } \mathbf{Z}_2 = \begin{pmatrix} l_2 & 0 \\ u_2 & l_2 + c_2 \end{pmatrix}$$

where $l_1, l_2 \in \mathfrak{l}$, $u_1, u_2 \in U$ and $c_1, c_2 \in \mathbb{C}$. Then,

$$\mathbf{Z}_1 \mathbf{Z}_2^* = \begin{pmatrix} l_1 l_2^* & l_1 u_2^* \\ u_1 l_2^* & u_1 u_2^* + l_1 c_2^* + c_1 l_2^* + c_1 c_2^* \end{pmatrix}.$$

From page 271 of [12], we see that

$$\text{Tr ad } \mathbf{X} \text{ ad } \mathbf{Y} = 2n \text{Tr}(\mathbf{X}\mathbf{Y}) - 2\text{Tr}(\mathbf{X})\text{Tr}(\mathbf{Y})$$

for $\mathbf{X}, \mathbf{Y} \in \mathfrak{gl}(V)$.

Since the semisimple Lie algebra \mathfrak{l} in $\mathfrak{gl}(V)$ is contained in $\mathfrak{sl}(V)$ which is the traceless subalgebra of $\mathfrak{gl}(V)$,

$$\begin{aligned} \{\mathbf{Z}_1, \mathbf{Z}_2\} &= 2n \text{Tr}(\mathbf{Z}_1 \mathbf{Z}_2^*) - 2\text{Tr}(\mathbf{Z}_1)\text{Tr}(\mathbf{Z}_2^*) \\ &= 2n \text{Tr}(l_1 l_2^*) + 2n \text{Tr}(u_1 u_2^*) + 2n_\alpha n_\beta c_1 \cdot c_2^* \end{aligned}$$

where $n = \dim(V)$, $n_\alpha = \dim(V_\alpha)$, and $n_\beta = \dim(V_\beta)$. Hence, $\{\cdot, \cdot\}$ is a positive definite Hermitian inner product on \mathfrak{g} . □

Remark 2.3. We rescale the Hermitian inner product on \mathfrak{g} via division by $2n$ for $n = \dim(V)$ (respectively, rescale the Cartan-Killing form). Thus,

$$\{\mathbf{Z}_1, \mathbf{Z}_2\} = \text{Tr}(l_1 l_2^*) + \text{Tr}(u_1 u_2^*) + \frac{n_\alpha n_\beta}{n} c_1 \cdot c_2^*.$$

Then, for $E_{ij} \in V_\alpha^* \otimes V_\beta$ which is zero except ij -component or if we write a unit column vector e_i in the j -th entry, we see $\{E_{ij}, E_{kl}^*\} = \text{Tr}(E_{ij}, E_{kl}^*) = \delta_{jl} e_i \cdot e_k^* = \delta_{ik} \delta_{jl}$.

Let \mathfrak{l} be a semisimple Lie algebra with $\text{rank}(\mathfrak{l}) = m$. We fix a Cartan subalgebra \mathfrak{h} . Let Φ be a set of roots of \mathfrak{l} relative to \mathfrak{h} . The root space decomposition of \mathfrak{g} is

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ is the root space for $\alpha \in \Phi$.

Definition 2.4. Let $\Delta = \{\alpha_1, \dots, \alpha_m\}$ be a set of simple roots of \mathfrak{l} associated with the Cartan subalgebra \mathfrak{h} . We define the *characteristic element* E_{α_i} associated with $\alpha_i \in \Delta$ as

$$\alpha_j(E_{\alpha_i}) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i, \end{cases}$$

Then, we can construct a gradation $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ which is called a *gradation associated with E_{α_i}* as follows:

$$\begin{aligned} \mathfrak{l}_0 &= \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{l}_\alpha \oplus \mathfrak{l}_{-\alpha} \\ \mathfrak{l}_k &= \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{l}_\alpha \\ \mathfrak{l}_{-k} &= \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{l}_{-\alpha} \quad (k > 0), \end{aligned}$$

where $\Phi_k^+ = \{\alpha \in \Phi^+ | \alpha(E_{\alpha_i}) = k\}$.

Then, we can construct a gradation $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ which is called a *gradation associated with α_i* . In this case, by Lemma 3.8 of [24], $\bigoplus_{p < 0} \mathfrak{l}_p$ is a fundamental graded Lie algebra.

Example 2.5. Let L be a semisimple Lie group. Let P_{α_i} be a maximal parabolic subgroup of L associated with a simple root α_i . The Lie algebra \mathfrak{l} of L has a gradation $\bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ associated with α_i . Then, the tangent space of the homogeneous space L/P_{α_i} at each point is identified with $\bigoplus_{p < 0} \mathfrak{l}_p$ which is a fundamental graded Lie algebra.

Proposition 2.6. Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \text{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$ be the corresponding Lie algebra. Then, we can give a gradation of $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ such that the graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is identified with the tangent space of X at a point x where x is in the open G -orbit.

More precisely, let \mathfrak{l}_k and U_k be eigenspaces that have eigenvalue k under the action of $E_X := E_\alpha$. Then,

$$\mathfrak{l} = \bigoplus_{k=-\mu(\mathfrak{l})}^{\mu(\mathfrak{l})} \mathfrak{l}_k \quad \text{and} \quad U = \bigoplus_{k=-\mu(U)}^{\mu(U)} U_k$$

where $\mu(\mathfrak{l})$ and $\mu(U)$ are the largest numbers among the nonzero eigenvalues of the action of E_X on \mathfrak{l} and U , respectively. Now give the gradation on \mathfrak{g} by shifting the above decompositions as follows:

$$\begin{aligned} \mathfrak{g}_{-p} &= \mathfrak{l}_{-p} \text{ for } -p < -1 \\ \mathfrak{g}_{-1} &= \mathfrak{l}_{-1} + U_{-\mu(U)} \\ \mathfrak{g}_0 &= (\mathfrak{l}_0 + \mathbb{C}) \triangleright U_{-\mu(U)+1} \\ \mathfrak{g}_p &= \mathfrak{l}_p + U_{-\mu(U)+p+1} \text{ for } p \geq 1 \end{aligned}$$

such that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for $p, q \in \mathbb{Z}$.

To prove Proposition 2.6, we need to calculate the eigenvalues under E_α -actions.

Lemma 2.7. *Let $X = (L, \alpha, \beta)$ and E_α be the characteristic element associated with root α . The Lie algebra of automorphism $\text{aut}(X) = (\mathfrak{l} + \mathbb{C}) \triangleright U$ has eigenspace decomposition under the action of $E_X = E_\alpha$ (\mathbb{C} has zero eigenvalue). Let \mathfrak{l}_k and U_k be eigenspaces that have eigenvalue k .*

(1) $(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$ where $U = V(\pi_m)$. Let $E_X = E_{\alpha_{m-1}}$ then

$$\begin{aligned} \mathfrak{l}_{-2} + \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2, \\ U_{-\frac{m-1}{2}} &+ U_{-\frac{m-1}{2}+1} + \cdots + U_{\frac{m-1}{2}-1} + U_{\frac{m-1}{2}}, \end{aligned}$$

and $\dim U_{-\frac{m-1}{2}} = 2$.

(2) $(B_3, \alpha_1, \alpha_3)$ where $U = V(\pi_3)$; let $E_X = E_{\alpha_1}$ then

$$\begin{aligned} \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1, \\ U_{-\frac{1}{2}} &+ U_{\frac{1}{2}}, \end{aligned}$$

and $\dim U_{-\frac{1}{2}} = 4$.

(3) $(C_m, \alpha_m, \alpha_{m-1})$ where $U = V(\pi_1)$. Let $E_X = E_{\alpha_m}$ then

$$\begin{aligned} \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1, \\ U_{-\frac{1}{2}} &+ U_{\frac{1}{2}}, \end{aligned}$$

and $\dim U_{-\frac{1}{2}} = m$.

(4) $(C_m, \alpha_{i+1}, \alpha_i)$, $m > 2$, $i = 1, \dots, m-2$ where $U = V(\pi_1)$. Let $E_X = E_{\alpha_{i+1}}$ then

$$\begin{aligned} \mathfrak{l}_{-2} + \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2, \\ U_{-1} &+ U_0 + U_1, \end{aligned}$$

and $\dim U_{-1} = i + 1$.

- (5) $(F_4, \alpha_2, \alpha_3)$ where α_2 is a long root and $U = V(\pi_4)$. Let $E_X = E_{\alpha_2}$ then

$$\begin{aligned} \mathfrak{l}_{-3} + \mathfrak{l}_{-2} + \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2 + \mathfrak{l}_3, \\ U_{-2} &+ U_{-1} + U_0 + U_1 + U_2, \end{aligned}$$

and $\dim U_{-2} = 3$.

- (6) $(G_2, \alpha_2, \alpha_1)$ where $U = V(\pi_1)$. Let $E_X = E_{\alpha_2}$ then

$$\begin{aligned} \mathfrak{l}_{-2} + \mathfrak{l}_{-1} &+ \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2, \\ U_{-1} &+ U_0 + U_1, \end{aligned}$$

and $\dim U_{-1} = 2$.

Furthermore, \mathfrak{l}_k and U_k are irreducible L_0 -representations.

Proof. It is calculated with basis elements from [23] or [18]. \square

Let $\mathfrak{l}_- = \bigoplus_{p < 0} \mathfrak{l}_p$ and $U_- = U_{-\mu(U)}$. Then, $\mathfrak{m} = \mathfrak{l}_- + U_-$.

Proof of Proposition 2.6. Let \tilde{X} be the blowup of X along Z . Since the open G -orbit of X is isomorphic to the open G -orbit of \tilde{X} , it is sufficient to show that $T_x \tilde{X}$ is identified with $\mathfrak{m} = \mathfrak{l}_- + U_-$ for any x that is in the open G -orbit of \tilde{X} .

According to Theorem 2.1 and its proof, $G = (L \times \mathbb{C}^*)/C \ltimes U$, where U is a L -representation space and C is the centralizer. \tilde{X} is a projective bundle over the L -orbit $Y \cong L/P_\alpha$ such that U acts by translation on the fibers, where P_α is the parabolic subgroup of L associated with the root α . For any point $x \in Y$, the tangent directions of the L -action at x are naturally identified with $\mathfrak{l}_- \cong T_x Y$, and the other tangent directions are contained in U .

Hence, we assume that x is in Y which contained in the open G -orbit of \tilde{X} and choose the characteristic element E_α of \mathfrak{l} associated with the root α as E_X : the *grading element* of \mathfrak{g} . Let \mathfrak{l}_k and U_k be eigenspaces that have eigenvalue k under the action of E_X . Then, according to Lemma 2.7, we see

$$\mathfrak{l} = \bigoplus_{k=-\mu(\mathfrak{l})}^{\mu(\mathfrak{l})} \mathfrak{l}_k \text{ and } U = \bigoplus_{k=-\mu(U)}^{\mu(U)} U_k$$

where $\mu(\mathfrak{l})$ and $\mu(U)$ are the largest numbers among the nonzero eigenvalues of the action of E_X on \mathfrak{l} and on U , respectively.

Since U_k is an irreducible L_0 -module and we see that $[\mathfrak{l}_{-1}, U_k] = U_{k-1}$, if the tangent space of X at x contains U_k , it must contain U_{k-1} . We can easily check that the dimension $\dim X = \dim L/(P_\alpha \cap P_\beta) + 1$ equals $\dim L/P_\alpha + \dim U_-$ in all cases. Hence, if one give the gradation on \mathfrak{g} by shifting based on $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} + U_{-\mu(U)}$, then the tangent space $T_x X$ at x is identified with $\mathfrak{m} = \mathfrak{l}_- + U_-$. Since the gradation is given by E_X , it is clear that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for $p, q \in \mathbb{Z}$. \square

Let $\mathfrak{l}_{>0} = \mathfrak{l}_1 + \cdots + \mathfrak{l}_{\mu(l)}$ and $\mathfrak{l}_{\geq 0} = \mathfrak{l}_0 + \mathfrak{l}_{>0}$. Let $U_{\tilde{i}} =: U_{-\mu(U)+1+i}$ for $i = -1, \dots, l$ where $l = 2\mu(U) - 1$. For example, $U_{\tilde{-1}} = U_{-\mu(U)}$ which is U_- , $U_{\tilde{0}} = U_{-\mu(U)+1}$, and $U_{\tilde{l}} = U_{\mu(U)}$. Let $U_+ = \bigoplus_{1 \leq i \leq 2\mu(U)-1} U_{\tilde{i}}$, and $U_{\geq \tilde{0}} = U_{\tilde{0}} + U_+ = \bigoplus_{0 \leq i \leq 2\mu(U)-1} U_{\tilde{i}}$. Let $\mathfrak{g}_{\geq 0} = \mathfrak{l}_{\geq 0} + U_{\geq \tilde{0}} + \mathbb{C}$.

Lemma 2.8. *Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a graded Lie algebra given in Proposition 2.6. Then,*

- (1) $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is fundamental, i.e., $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$;
- (2) If $z \in \mathfrak{l}_{\geq 0} + U_{\geq \tilde{0}}$ satisfies $[z, \mathfrak{l}_{-1}] = 0$, then $z = 0$. Further, if $z \in \mathfrak{g}_{\geq 0}$ satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $z = 0$;
- (3) for any nonzero vector $u \in U_{\tilde{0}}$, the dimension of the subspace $[\mathfrak{l}_{-1}, u] \subset U_-$ is greater than or equal to 2.

Proof. (1) The gradation of $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ associated with α satisfies $\mathfrak{l}_p = [\mathfrak{l}_{p+1}, \mathfrak{l}_{-1}]$ for $p < -1$. For $p = -2$, we see

$$\begin{aligned} \mathfrak{g}_{-2} = \mathfrak{l}_{-2} &= [\mathfrak{l}_{-1}, \mathfrak{l}_{-1}] \\ &= [\mathfrak{l}_{-1} + U_-, \mathfrak{l}_{-1} + U_-] \text{ because } [\mathfrak{l}_{-1} + U_-, U_-] = 0 \\ &= [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]. \end{aligned}$$

For $p < -2$, we see

$$\begin{aligned} \mathfrak{g}_p = \mathfrak{l}_p &= [\mathfrak{l}_{p+1}, \mathfrak{l}_{-1}] \\ &= [\mathfrak{l}_{p+1}, \mathfrak{l}_{-1} + U_-] \text{ because } [\mathfrak{l}_{p+1}, U_-] = 0 \\ &= [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]. \end{aligned}$$

Hence, the graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is fundamental.

- (2) Assume $z \in U_{\geq \tilde{0}}$ satisfies $[z, \mathfrak{l}_{-1}] = 0$. Since $[U_{\widetilde{k-1}}, \mathfrak{l}_1] = U_{\tilde{k}}$ for $k \geq 0$,

$$0 = \{[z, \mathfrak{l}_{-1}], U_{\widetilde{k-1}}\} = \{z, [\mathfrak{l}_1, U_{\widetilde{k-1}}]\} = \{z, U_{\tilde{k}}\},$$

which implies that $z = 0$. By Lemma 1.3 of [22], if $z \in \mathfrak{l}_{\geq 0}$ satisfies $[z, \mathfrak{l}_{-1}] = 0$, then $z = 0$. Hence, if $z \in \mathfrak{l}_{\geq 0} + U_{\geq \tilde{0}}$ satisfies $[z, \mathfrak{l}_{-1}] = 0$, then $z = 0$.

Since $\mathfrak{l}_{-1} \subset \mathfrak{g}_{-1}$, if $z \in \mathfrak{l}_{\geq 0} + U_{\geq \tilde{0}}$ satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $z = 0$.

If $z \in \mathbb{C}$ satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $[z, U_-] = z.U_- = 0$ because $[\mathbb{C}, \mathfrak{l}_{-1}] = 0$. For $z \in \mathbb{C}$, if $z.U_- = 0$, then $z = 0$. Hence, if $z \in \mathfrak{g}_{\geq 0} = \mathfrak{l}_{\geq 0} + U_{\geq \tilde{0}} + \mathbb{C}$ satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $z = 0$.

(3) The action

$$\begin{aligned} \mathfrak{l}_{-1} \times U_{\tilde{0}} &\rightarrow U_- \\ (l, u) &\mapsto [l, u] \end{aligned}$$

is described as following list up to scalar. The following list is from weights and weight diagrams([4]) of the irreducible \mathfrak{l}_0 -representations on \mathfrak{l}_k and U_k . Let $R\omega(T)$ be the irreducible representation of type T with the highest weight ω . Let $*$ be the usual complex conjugation.

- (a) $(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$, where $U = V(\pi_m)$. Let $R\pi_1(A_1) = W$ be the standard representation of A_1 . Let $R\pi_1(A_{m-2}) = Q$ be the standard representation of A_{m-2} . Then, $\dim W = 2$, $W^* = W$, $\dim Q = m - 1$ and

$$\mathfrak{l}_{-1} = R\pi_1(A_{m-2})^* \otimes R\pi_1(A_1)^* = Q^* \otimes \text{Sym}^2 W^*$$

$$U_- = R\pi_1(A_1)^* = W^*$$

$$U_{\tilde{0}} = R\pi_1(A_{m-2}) \otimes R\pi_1(A_1) = Q \otimes W.$$

The action $\mathfrak{l}_{-1} \times U_{\tilde{0}} \rightarrow U_-$ is given as follows, for $w_1, w_2 \in W$ such that $W = \langle w_1, w_2 \rangle$ and $q \in Q$:

$$(Q^* \otimes \text{Sym}^2 W^*) \times (Q \otimes W) \rightarrow W^*$$

$$(q^* \otimes w_1^* \odot w_2^*, q \otimes w_1) \mapsto q^*(q)w_1^* \odot w_2^*(w_1) = w_2^*$$

$$(q^* \otimes w_1^* \odot w_1^*, q \otimes w_1) \mapsto q^*(q)w_1^* \odot w_1^*(w_1) = 2w_1^*.$$

- (b) $(B_3, \alpha_1, \alpha_3)$ where $U = V(\pi_3)$. Let W be the spin representation of B_2 . Let V be the standard representation of B_2 . Then, $V^* = V$, $\dim V = 5$, $\dim W = 4$, $W = W^*$ and

$$\mathfrak{l}_{-1} = R\pi_1(B_2)^* = V^*$$

$$U_- = R\pi_2(B_2)^* = W^*$$

$$U_{\tilde{0}} = R\pi_2(B_2) = W.$$

The action $\mathbb{L}_{-1} \times U_{\bar{0}} \rightarrow U_-$ is given by the following:

\times	v_1^*	v_2^*	v_3^*	v_4^*	v_5^*
w_1	w_4^*	w_3^*	w_2^*	\cdot	\cdot
w_2	w_3^*	\cdot	w_1^*	w_4^*	\cdot
w_3	w_2^*	w_1^*	\cdot	\cdot	w_4^*
w_4	w_1^*	\cdot	\cdot	w_3^*	w_2^*

where $\{w_1, w_2, w_3, w_4\}$ is a basis of W and $\{v_1, v_2, v_3, v_4, v_5\}$ is a basis of V .

- (c) $(C_m, \alpha_m, \alpha_{m-1})$ where $U = V(\pi_1)$. Let W be the standard representation of A_{m-1} . Then, $\dim W = m$ and

$$\mathbb{L}_{-1} = R2\pi_1(A_{m-1})^* = \text{Sym}^2 W^*$$

$$U_- = R\pi_1(A_{m-1})^* = W^*$$

$$U_{\bar{0}} = R\pi_1(A_{m-1}) = W.$$

The action $\mathbb{L}_{-1} \times U_{\bar{0}} \rightarrow U_-$ is given as follows, for the orthonormal basis $w_i, w_j, w_k \in W$:

$$\text{Sym}^2 W^* \times W \rightarrow W^*$$

$$(w_i^* \odot w_j^*, w_k) \mapsto (w_i^* \odot w_j^*)(w_k) = \delta_{jk} w_i^* + \delta_{ik} w_j^*.$$

- (d) $(C_m, \alpha_{i+1}, \alpha_i)$, $m > 2$, $i = 1, \dots, m-2$, where $U = V(\pi_1)$. Let W be the standard representation of A_i and let Q be the standard representation of C_{m-i-1} . Then, $\dim W = i+1$, $\dim Q = 2m-2i-2$ and

$$\mathbb{L}_{-1} = R\pi_1(A_i)^* \otimes R\pi_1(C_{m-i-1})^* = W^* \otimes Q^*$$

$$U_- = R\pi_1(A_i)^* = W^*$$

$$U_{\bar{0}} = R\pi_1(C_{m-i-1}) = Q.$$

The action $\mathbb{L}_{-1} \times U_{\bar{0}} \rightarrow U_-$ is given as follows, for $q \in Q$ and $w \in W$:

$$(W^* \otimes Q^*) \times Q \rightarrow W^*$$

$$(w^* \otimes q^*, q) \mapsto w^* q^*(q).$$

- (e) $(F_4, \alpha_2, \alpha_3)$ where α_2 is a long root and $U = V(\pi_4)$. Let W be the standard representation of A_1 and let V be the standard representation of A_2 . Then, $\dim V = 3$, $\dim W = 2$, $W^* = W$

and

$$\mathfrak{l}_{-1} = R2\pi_1(A_2) \otimes R\pi_1(A_1) = \text{Sym}^2 V \otimes W$$

$$U_- = R\pi_1(A_2) = V$$

$$U_{\tilde{0}} = R\pi_1(A_2)^* \otimes R\pi_1(A_1)^* = V^* \otimes W^*.$$

By the action $\mathfrak{l}_{-1} \times U_{\tilde{0}} \rightarrow U_-$, for $v_1, v_2, v_3 \in V$ and $w \in W$,

$$(\text{Sym}^2 V \otimes W) \times (V^* \otimes W^*) \rightarrow V$$

$$(v_1 \odot v_1 \otimes w, v_1^* \otimes w^*) \mapsto v_1 \odot v_1(v_1^*)w(w^*) = 2v_1$$

$$(v_1 \odot v_2 \otimes w, v_1^* \otimes w^*) \mapsto v_1 \odot v_2(v_1^*)w(w^*) = v_2$$

$$(v_1 \odot v_3 \otimes w, v_1^* \otimes w^*) \mapsto v_1 \odot v_3(v_1^*)w(w^*) = v_3.$$

(f) $(G_2, \alpha_2, \alpha_1)$ where $U = V(\pi_1)$. Let W be the standard representation of A_1 . Then, $\dim W = 2$, $W^* = W$ and

$$\mathfrak{l}_{-1} = R3\pi_1(A_1)^* = \text{Sym}^3 W^*$$

$$U_- = R\pi_1(A_1)^* = W^*$$

$$U_{\tilde{0}} = R2\pi_1(A_1) = \text{Sym}^2 W.$$

By the action $\mathfrak{l}_{-1} \times U_{\tilde{0}} \rightarrow U_-$, for $w_1, w_2 \in W$,

$$\text{Sym}^3 W^* \times \text{Sym}^2 W \rightarrow W^*$$

$$(w_1^* \odot w_2^* \odot w_1^*, w_1 \odot w_2) \mapsto w_1^* \odot w_2^* \odot w_1^*(w_1 \odot w_2) = 2w_1^*$$

$$(w_1^* \odot w_2^* \odot w_2^*, w_1 \odot w_2) \mapsto w_1^* \odot w_2^* \odot w_2^*(w_1 \odot w_2) = 2w_2^*$$

$$(w_1^* \odot w_1^* \odot w_1^*, w_1 \odot w_1) \mapsto w_1^* \odot w_1^* \odot w_1^*(w_1 \odot w_1) = 3w_1^*$$

$$(w_1^* \odot w_1^* \odot w_2^*, w_1 \odot w_1) \mapsto w_1^* \odot w_1^* \odot w_2^*(w_1 \odot w_1) = 2w_2^*.$$

From the above list of actions $\mathfrak{l}_{-1} \times U_{\tilde{0}} \rightarrow U_-$, we easily see that for a nonzero vector $u \in U_{\tilde{0}}$, the dimension of the subspace $[\mathfrak{l}_{-1}, u] \subset U_-$ is greater than or equal to 2.

□

3. VANISHING COHOMOLOGIES

Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \text{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$ be the corresponding Lie algebra. By Proposition 2.6, we can give a gradation of $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ such that the graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is identified with the tangent space of X at a point x , where x is in the open

G -orbit. Let $\mathfrak{l}_- = \bigoplus_{p < 0} \mathfrak{l}_p$, $\mathfrak{l}_{>0} = \mathfrak{l}_1 + \cdots + \mathfrak{l}_{\mu(\mathfrak{l})}$ and $\mathfrak{l}_{\geq 0} = \mathfrak{l}_0 + \mathfrak{l}_{>0}$. Let $U_i =: U_{-\mu(U)+1+i}$ for $i = -1, \dots, l$ where $l = 2\mu(U) - 1$. Let $U_- = U_{\leq -1}$, $U_+ = \bigoplus_{1 \leq i \leq 2\mu(U)-1} U_i = U_{-\mu(U)+2} + \cdots + U_{\mu(U)}$, and $U_{\geq \bar{0}} = U_{\bar{0}} + U_+$. Let $\mathfrak{m} = \mathfrak{l}_- + U_-$, $\mathfrak{g}_{>0} = \mathfrak{l}_{>0} + U_+$, and $\mathfrak{g}_{\geq 0} = \mathfrak{l}_{\geq 0} + U_{\geq \bar{0}} + \mathbb{C}$, and let \mathfrak{m}' be the dual of \mathfrak{m} .

Proposition 3.1. *Let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$ and $\mathfrak{m} = \mathfrak{l}_- + U_-$. Assume that*

- (1) *if $z \in \mathfrak{l}_{\geq 0} + U_{\geq \bar{0}}$ satisfies $[\mathfrak{l}_-, z] = 0$, then $z = 0$;*
- (2) *for any vector $u \in U_{\bar{0}}$, if the dimension of the subspace $[\mathfrak{l}_{-1}, u] \subset U_-$ is less than or equal to 1, then $u = 0$.*

For $p > 0$, if $H^{p,1}(\mathfrak{l}_-, \mathfrak{l}) = 0$ and $H^{p,1}(\mathfrak{l}_-, U) = 0$, then $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$.

Proof. For $p > 0$, $C^{p,1}(\mathfrak{m}, \mathfrak{g}) \subset \mathfrak{m}' \otimes \mathfrak{g}$. Let $\phi \in C^{p,1}(\mathfrak{m}, \mathfrak{g})$ such that $\partial\phi = 0$. We will show that there exist $\psi \in \mathfrak{g}$ such that

$$\partial\psi = \phi.$$

Write $\phi = \phi_{\mathfrak{l}} + \phi_{\mathbb{C}} + \phi_U$, where $\phi_{\mathfrak{l}} \in \mathfrak{m}' \otimes \mathfrak{l}$, $\phi_{\mathbb{C}} \in \mathfrak{m}' \otimes \mathbb{C}$, and $\phi_U \in \mathfrak{m}' \otimes U$.

For any $X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-} \in \mathfrak{l}_-$ and $X^{U_-}, Y^{U_-} \in U_-$, we have

$$\begin{aligned} 0 &= \partial\phi(X^{\mathfrak{l}_-} + X^{U_-}, Y^{\mathfrak{l}_-} + Y^{U_-}) \\ &= [X^{\mathfrak{l}_-} + X^{U_-}, \phi(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{\mathfrak{l}_-} + Y^{U_-}, \phi(X^{\mathfrak{l}_-} + X^{U_-})] - \phi([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \\ &\quad \text{because } [\mathfrak{l}_- + U_-, U_-] = 0 \\ &= \left\{ [X^{\mathfrak{l}_-}, \phi_{\mathfrak{l}+\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{\mathfrak{l}_-}, \phi_{\mathfrak{l}+\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-})] - \phi_{\mathfrak{l}+\mathbb{C}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \right\} \\ &\quad + \left\{ [X^{U_-}, \phi_{\mathfrak{l}+\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{U_-}, \phi_{\mathfrak{l}+\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-})] \right. \\ &\quad \left. + [X^{\mathfrak{l}_-}, \phi_U(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{\mathfrak{l}_-}, \phi_U(X^{\mathfrak{l}_-} + X^{U_-})] - \phi_U([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &\stackrel{(*)}{=} \phi_{\mathbb{C}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \\ 0 &\stackrel{(*)}{=} [X^{\mathfrak{l}_-}, \phi_{\mathfrak{l}}(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{\mathfrak{l}_-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}_-} + X^{U_-})] - \phi_{\mathfrak{l}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \\ 0 &\stackrel{(\otimes)}{=} [X^{U_-}, \phi_{\mathfrak{l}}(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{U_-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}_-} + X^{U_-})] \\ &\quad + [X^{U_-}, \phi_{\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{U_-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-})] \\ &\quad + [X^{\mathfrak{l}_-}, \phi_U(Y^{\mathfrak{l}_-} + Y^{U_-})] - [Y^{\mathfrak{l}_-}, \phi_U(X^{\mathfrak{l}_-} + X^{U_-})] - \phi_U([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]). \end{aligned}$$

Put $X^{U_-} = Y^{U_-} = 0$ into $(*)$ to get

$$(3.2) \quad [X^{\mathfrak{l}_-}, \phi_{\mathfrak{l}}(Y^{\mathfrak{l}_-})] - [Y^{\mathfrak{l}_-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}_-})] - \phi_{\mathfrak{l}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) = 0.$$

Put $Y^{\mathfrak{l}-} = 0$ into $(*)$ to get

$$(3.3) \quad [X^{\mathfrak{l}-}, \phi_{\mathfrak{l}}(Y^{U-})] = 0.$$

Put $Y^{\mathfrak{l}-} = 0$ and $X^{U-} = 0$ into (\diamond) to get

$$(3.4) \quad [X^{\mathfrak{l}-}, \phi_U(Y^{U-})] - [Y^{U-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}-})] - [Y^{U-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}-})] = 0.$$

Put $Y^{\mathfrak{l}-} = 0$ and $X^{\mathfrak{l}-} = 0$ into (\diamond) to get

$$(3.5) \quad [X^{U-}, \phi_{\mathfrak{l}}(Y^{U-})] - [Y^{U-}, \phi_{\mathfrak{l}}(X^{U-})] \\ + [X^{U-}, \phi_{\mathbb{C}}(Y^{U-})] - [Y^{U-}, \phi_{\mathbb{C}}(X^{U-})] = 0.$$

Put $X^{U-} = Y^{U-} = 0$ into (\diamond) to get

$$(3.6) \quad [X^{\mathfrak{l}-}, \phi_U(Y^{\mathfrak{l}-})] - [Y^{\mathfrak{l}-}, \phi_U(X^{\mathfrak{l}-})] - \phi_U([X^{\mathfrak{l}-}, Y^{\mathfrak{l}-}]) = 0.$$

By (3.2) and (3.6), we have $\partial(\phi_{\mathfrak{l}} + \phi_U)(X^{\mathfrak{l}-}, Y^{\mathfrak{l}-}) = 0$. By hypothesis, $H^{p,1}(\mathfrak{l}_-, \mathfrak{l}) = 0$ and $H^{p,1}(\mathfrak{l}_-, U) = 0$ for $p > 0$. Then, there exist $\psi = \psi_{\mathfrak{l}} + \psi_U$, where $\psi_{\mathfrak{l}} \in \mathfrak{l}$ and $\psi_U \in U$ such that

$$(3.7) \quad \partial\psi(X^{\mathfrak{l}-}) = (\phi_{\mathfrak{l}} + \phi_U)(X^{\mathfrak{l}-}).$$

In (3.3), since $X^{\mathfrak{l}-} \in \mathfrak{l}_-$ is arbitrary, by assumption (1), we have

$$(3.8) \quad \phi_{\mathfrak{l}}(Y^{U-}) = 0.$$

In (3.5), by (3.8), we see that

$$(3.9) \quad [X^{U-}, \phi_{\mathbb{C}}(Y^{U-})] - [Y^{U-}, \phi_{\mathbb{C}}(X^{U-})] = 0.$$

Equation (3.9) is also valid for the two linearly independent vectors X^{U-} and Y^{U-} , and \mathbb{C} act on U as scalars. Hence,

$$(3.10) \quad \phi_{\mathbb{C}}(X^{U-}) = 0.$$

We will show that $\partial\psi = \phi$ where $\phi = \phi_{\mathfrak{l}} + \phi_U + \phi_{\mathbb{C}}$. By (\star) , (3.7), (3.8), and (3.10), it suffices to show that $\phi_{\mathbb{C}}(X^{\mathfrak{l}-}) = 0$ for all $X^{\mathfrak{l}-} \in \mathfrak{l}_{-1}$ and $\partial\psi(X^{U-}) = \phi_U(X^{U-})$ for all $X^{U-} \in U_-$.

Let $X^{\mathfrak{l}-} \in \mathfrak{l}_{-1}$. In (3.4), we have

$$\begin{aligned} [Y^{U-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}-})] &= [X^{\mathfrak{l}-}, \phi_U(Y^{U-})] - [Y^{U-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}-})] \\ &= [X^{\mathfrak{l}-}, \phi_U(Y^{U-})] - [Y^{U-}, [X^{\mathfrak{l}-}, \psi_{\mathfrak{l}}]] \text{ because } \phi_{\mathfrak{l}}(X^{\mathfrak{l}-}) = \partial\psi_{\mathfrak{l}}(X^{\mathfrak{l}-}) \\ &= [X^{\mathfrak{l}-}, \phi_U(Y^{U-})] - [X^{\mathfrak{l}-}, [Y^{U-}, \psi_{\mathfrak{l}}]] \text{ because } [\mathfrak{l}_-, U_-] = 0 \\ &= [X^{\mathfrak{l}-}, (\phi_U(Y^{U-}) - [Y^{U-}, \psi_{\mathfrak{l}}])]. \end{aligned}$$

Hence, for $X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1}$ and $Y^{U_-} \in U_-$,

$$(3.11) \quad [Y^{U_-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}_-})] = [X^{\mathfrak{l}_-}, \phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-})].$$

Because \mathbb{C} act on U as a scalar, left side of (3.11) is in U_- and hence, the $\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-})$ in the bracket of right side is in $U_{\tilde{0}}$

From the decomposition $\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}) = \bigoplus_i (\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{i}}$, where $(\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{i}} \in U_{\tilde{i}}$, we have

$$(3.12) \quad \begin{aligned} [Y^{U_-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}_-})] &= [X^{\mathfrak{l}_-}, (\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{0}}] \\ 0 &= [X^{\mathfrak{l}_-}, (\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{i}}] \text{ for } \tilde{i} \neq \tilde{0}. \end{aligned}$$

For $X^{\mathfrak{l}_-} \in \{X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1} | \phi_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0\}$,

$$0 = [X^{\mathfrak{l}_-}, (\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{0}}].$$

Since $\{X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1} | \phi_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0\} \subset \mathfrak{l}_{-1}$ is a hyperplane or \mathfrak{l}_- , by assumption (2),

$$(3.13) \quad 0 = (\phi_U(Y^{U_-}) - \partial\psi_{\mathfrak{l}}(Y^{U_-}))_{\tilde{0}}.$$

By (3.12) and (3.13), for any $Y^{U_-} \in U_-$ and $X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1}$,

$$[Y^{U_-}, \phi_{\mathbb{C}}(X^{\mathfrak{l}_-})] = 0.$$

Hence, for any $X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1}$,

$$(3.14) \quad \phi_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0.$$

By (3.14), equation (3.4) becomes

$$(3.15) \quad [X^{\mathfrak{l}_-}, \phi_U(Y^{U_-})] = [Y^{U_-}, \phi_{\mathfrak{l}}(X^{\mathfrak{l}_-})] = [Y^{U_-}, \partial\psi_{\mathfrak{l}}(X^{\mathfrak{l}_-})].$$

Write $\partial\psi = (\partial\psi)_{\mathfrak{l}} + (\partial\psi)_{\mathbb{C}} + (\partial\psi)_U$ where $(\partial\psi)_{\mathfrak{l}} \in \mathfrak{m}' \otimes \mathfrak{l}$, $(\partial\psi)_{\mathbb{C}} \in \mathfrak{m}' \otimes \mathbb{C}$ and $(\partial\psi)_U \in \mathfrak{m}' \otimes U$. Then,

$$\begin{aligned} \partial\psi(X^{\mathfrak{l}_-}) &= [X^{\mathfrak{l}_-}, \psi] = [X^{\mathfrak{l}_-}, \psi_{\mathfrak{l}}] + [X^{\mathfrak{l}_-}, \psi_U] \\ \partial\psi(X^{U_-}) &= [X^{U_-}, \psi] = [X^{U_-}, \psi_{\mathfrak{l}}]. \end{aligned}$$

Thus,

$$(3.16) \quad (\partial\psi)_{\mathbb{C}} = 0$$

$$(3.17) \quad (\partial\psi)_{\mathfrak{l}}(X^{U_-}) = 0,$$

and

$$\begin{aligned} (\partial\psi)_\mathfrak{l}(X^{\mathfrak{l}-}) &= [X^{\mathfrak{l}-}, \psi_\mathfrak{l}] \\ (\partial\psi)_U(X^{\mathfrak{l}-}) &= [X^{\mathfrak{l}-}, \psi_U] \\ (\partial\psi)_U(X^{U-}) &= [X^{U-}, \psi_\mathfrak{l}]. \end{aligned}$$

In particular, this implies that

$$\begin{aligned} [X^{\mathfrak{l}-}, (\partial\psi)_U(X^{U-})] &= [X^{\mathfrak{l}-}, [X^{U-}, \psi_\mathfrak{l}]] \\ &= [X^{U-}, [X^{\mathfrak{l}-}, \psi_\mathfrak{l}]] \text{ because } [\mathfrak{l}_-, U_-] = 0 \\ &= [X^{U-}, (\partial\psi)_\mathfrak{l}(X^{\mathfrak{l}-})]. \end{aligned}$$

Hence, by (3.15),

$$(3.18) \quad [X^{\mathfrak{l}-}, (\partial\psi)_U(X^{U-})] = [X^{\mathfrak{l}-}, \phi_U(X^{U-})].$$

Since $X^{\mathfrak{l}-} \in \mathfrak{l}_-$ is arbitrary in (3.18), by assumption (1), we have

$$(\partial\psi)_U(X^{U-}) = \phi_U(X^{U-}).$$

Hence, by (3.16) and (3.17),

$$(3.19) \quad (\partial\psi)(X^{U-}) = \phi_U(X^{U-}).$$

It follows that $\partial\psi = \phi$. Therefore, $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for any positive integer p . \square

Lemma 3.20. $H^{p,1}(\mathfrak{l}_-, U) = 0$ for $p > 0$.

Proof. Let $z_\alpha \in \mathfrak{l}$ be a root vector associated with the simple root α . Let σ_α be the simple reflection associated with α . Let λ be the highest weight of \mathfrak{l} on the irreducible representation U . Then, $-\lambda$ is the lowest weight on U . Let $u_{-\sigma_\alpha(\lambda)} \in U$ be the weight vector with weight $-\sigma_\alpha(\lambda)$.

By Theorem 5.15 of [14], we have

$$H^1(\mathfrak{l}_-, U) = \mathcal{H}^{\xi_\sigma},$$

where \mathcal{H}^{ξ_σ} is the irreducible \mathfrak{l}_0 -module with the lowest weight vector $z'_\alpha \otimes u_{-\sigma_\alpha(\lambda)}$ having weight $\xi_\sigma = -(\sigma_\alpha(\lambda) + \alpha)$.

Since $z'_\alpha \otimes u_{-\sigma_\alpha(\lambda)} \in \mathfrak{l}'_{-1} \otimes U_-$ and $\mathcal{H}^{\xi_\sigma} = \mathfrak{l}'_{-1} \otimes U_- = H^{0,1}(\mathfrak{l}_-, U)$, we see that $H^{p,1}(\mathfrak{l}_-, U) = 0$ for $p > 0$. \square

Proposition 3.21. Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a graded Lie algebra given in Proposition 2.6. $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p > 0$.

Proof. By Lemma of [24], $H^{p,1}(\mathbb{L}, \mathbb{I}) = 0$ for $p > 0$. By Lemma 3.20, $H^{p,1}(\mathbb{L}, U) = 0$ for $p > 0$. By (2) and (3) of Lemma 2.8, the two assumptions of Proposition 3.1 are satisfied. Hence, $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p > 0$. \square

Proposition 3.22. *Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a graded Lie algebra given in Proposition 2.6. The Lie algebra \mathfrak{g} is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.*

Proof. By Lemma 2.8, $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is fundamental, i.e., $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$ and if $z \in \mathfrak{g}_p$ for $p \geq 0$ satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $z = 0$. By Proposition 3.21, $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p \geq 1$. Hence, according to Lemma 1.4, the graded Lie algebra $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ of Proposition 2.6 is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. \square

4. GEOMETRIC STRUCTURES

Let X be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let X° be the open orbit of X with respect to $G = \text{Aut}(X)$. Let \mathfrak{g} be the Lie algebra of G . We recall, from Proposition 2.6, that there is a gradation of \mathfrak{g} such that $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is fundamental and

$$(4.1) \quad \iota : T_x X^\circ \simeq \mathfrak{m}$$

for a base point $x \in X^\circ$.

Definition 4.2. Let (X°, E) be the regular differential system of type \mathfrak{m} derived from the subbundle E of TX° , where E_x corresponds to \mathfrak{g}_{-1} under the identification $T_x X^\circ \simeq \mathfrak{m}$ for a base point $x \in X^\circ$. Let $G_0 \subset G_0(\mathfrak{m})$ be the Lie subgroup corresponding to \mathfrak{g}_0 . Let \mathcal{R} be the frame bundle of (X°, E) . Then, \mathcal{R} is isomorphic to $G \times_H G_0(\mathfrak{m})$, where $H = H(\mathfrak{m}, G_0)$.

The G_0 -subbundle \mathcal{P} of \mathcal{R} , which is isomorphic to the G_0 -subbundle $G \times_H G_0$ of $G \times_H G_0(\mathfrak{m})$, is a G_0 -structure on (X°, E) referred to as *the standard geometric structure* on X .

Definition 4.3. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be the fundamental graded Lie algebra given by Proposition 2.6, which satisfies (4.1). Let M be a projective manifold. A *distribution* D is a subbundle of $T(M)$, which is defined on outside of a subvariety $\text{Sing}(D)$ of M . Suppose that

- (1) any point $x \in M^\circ := M - \text{Sing}(D)$ is general and $\text{Sing}(D)$ has codimension at least two;
- (2) (M°, D) is a regular filtered manifold derived from a regular differential system of type \mathfrak{m} .

A G_0 -structure on (M°, D) is called a *geometric structure of M modeled on X* (See Definition 1.6).

Two geometric structures of M_1 and M_2 modeled on X are *locally equivalent* if the G_0 -structure on (M_1°, D_1) and the G_0 -structure on (M_2°, D_2) are locally equivalent. A geometric structure modeled on X is *locally flat* if it is locally equivalent to the standard geometric structure on X .

Proposition 4.4. *Let X be a smooth nonhomogeneous horospherical variety (L, α, β) of Picard number one. Let $G = \text{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$ be the corresponding Lie algebra. As in Proposition 2.6, we give a gradation on the Lie algebra \mathfrak{g} . Let $H := H(\mathfrak{m}, G_0)$ be the Lie subgroup of G associated with $\mathfrak{h} = \bigoplus_{p \geq 0} \mathfrak{g}_p$ (See (1.12)). Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ and let G_0 be the Lie subgroup of G corresponding to \mathfrak{g}_0 . Let (M, F) be a regular filtered manifold of type \mathfrak{m} . Then, for a given G_0 -structure on (M, F) , there exists a Cartan connection of type (\mathfrak{g}, H) so that two G_0 -structures on (M, F) are (locally) equivalent when the associated Cartan connections of type (\mathfrak{g}, H) are (locally) equivalent.*

Proof. We will apply Theorem 1.15 to $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \triangleright U$. As in Proposition 2.6, the element $E_X = E_\alpha \in \mathfrak{g}_0$ gives the gradation of \mathfrak{g} . By Lemma 2.8 (1), \mathfrak{m} is a fundamental graded Lie subalgebra. By Proposition 3.22, the Lie algebra \mathfrak{g} is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

By Proposition 2.2, let $\tilde{\mathfrak{g}} = \mathfrak{gl}(V)$ which contains \mathfrak{g} and \mathfrak{g}^* . The Killing form on $\mathfrak{gl}(V)$ itself is an $\text{ad}(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ -invariant symmetric bilinear form (\cdot, \cdot) such that the restricted inner product $\{\cdot, \cdot\}$ on \mathfrak{g} is a positive definite Hermitian inner product. This proves (1) of Theorem 1.15.

We can give a gradation on $\mathfrak{gl}(V)$ by the element E_X . Since V is a representation space of \mathfrak{l} , we have decomposition $\mathfrak{gl}(V) = \bigoplus_k \mathfrak{gl}(V)_k$, where $\mathfrak{gl}(V)_k$ is eigenspace of E_X with eigenvalue k . We shift this decomposition to give a gradation on $\mathfrak{gl}(V)$ satisfying (2) of Theorem 1.15.

Since $U \subset V_\alpha \otimes V_\beta^*$, we shift the gradation on $V_\alpha \otimes V_\beta^*$ to make it the extended gradation of the gradation on U . Then, $\mathfrak{g}_p \subset \tilde{\mathfrak{g}}_p$ for any integer p .

We also shift the gradation on U^* and extend it to $V_\alpha^* \otimes V_\beta$ such that $\tau(\mathfrak{g}_p) \subset \tilde{\mathfrak{g}}_{-p}$ for $p \geq 0$. More precisely, we can do this as follows: We have $\tau(E_X) = -E_X$, and for $z_1, z_2 \in \mathfrak{g}$, $[\tau(z_1), \tau(z_2)] = \tau([z_1, z_2])$. Hence, for $z \in \mathfrak{g}_p$ such that $[E_X, z] = kz$ where $k \in \mathbb{Z}$, we see that

$$[E_X, \tau(z)] = -[\tau(E_X), \tau(z)] = -\tau([E_X, z]) = -k\tau(z).$$

In Proposition 2.6, if one shifted gradation on U_k by j , i.e. $p = k + j$, then one shift gradation on $U_{-k}^* := (U_k)^*$ by $-j$, and also shift gradation on $V_\alpha^* \otimes V_\beta$ by $-j$ en bloc.

The remaining conditions (3) and (4) of Theorem 1.15 are clear. \square

The next proposition is proved in [11] providing the essence of Theorem 4.1 in [2].

Proposition 4.5 (From Proposition 2.9 of [11]). *Let M be a manifold. Assume that there exists a non-constant holomorphic map $f: \mathbb{P}^1 \rightarrow M$ such that $f^*T(M)$ is a positive vector bundle, i.e., $f^*T(M) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$ where $a_i \geq 1$ and $n = \dim M$. Let H be a closed connected subgroup of a connected Lie group G . Let \mathfrak{g} be the Lie algebra of G . Then, any Cartan connection on M of type (\mathfrak{g}, H) is locally flat.*

Sketch of the proof. Given a Cartan connection ω on a principal H -bundle $P \rightarrow M$, we can associate a principal G -bundle $\tilde{P} \rightarrow M$ with an Ehresmann connection $\tilde{\omega}$ as in Section 3 of [2]. For a curve $f_t: \mathbb{P}^1 \rightarrow M$ with positive $f_t^*T(M)$, we see that $f_t^*K(\tilde{\omega}) = 0$, where $K(\tilde{\omega})$ is the curvature of the connection $\tilde{\omega}$. We can see the vanishing of that curvature along a curve with positive tangents in the proof of Theorem 3.1 in [1].

By assumption, there is a non-constant holomorphic map $f: \mathbb{P}^1 \rightarrow M$ such that $f^*T(M)$ is a positive vector bundle. Then, there exists a family of holomorphic maps

$$\{f_t: \mathbb{P}^1 \rightarrow M | t \in \Delta^k, f_t^*T(M) \text{ positive}\}$$

parametrized by a polydisc Δ^k , for some $k > 0$ such that the union of their images $\cup_{t \in \Delta^k} f_t(\mathbb{P}^1)$ contains a nonempty open subset U of M . For a nonempty open set $U \subset M$, the curvature $K(\tilde{\omega})$ vanishes on U as stated above and hence vanishes on the whole space M . Thus, $\tilde{\omega}$ is locally flat on M , which implies that ω is locally flat on M . \square

The following is from Proposition 7.9 of [11], which is well known from Proposition II.3.7 and Theorem IV.3.7 of [13].

Proposition 4.6. *Let M be a uniruled projective manifold of Picard number one. Then, for any subvariety $Z \subset M$ of codimension two, there exists $f: \mathbb{P}^1 \rightarrow M$ with $f(\mathbb{P}^1) \cap Z = \emptyset$ and $f^*T(M)$ is positive.*

Theorem 4.7. *Let X be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let M be a Fano manifold of Picard number*

one. Then, any geometric structure of M modeled on X is locally equivalent to the standard geometric structure on X .

Proof. Let M be a Fano manifold of Picard number one. A G_0 -structure on (M^o, D) is given by the geometric structure of M modeled on X , where D is a subbundle of $T(M)$ with singularity $\text{Sing}(D)$ and $M^o = M - \text{Sing}(D)$. By Proposition 4.4, that regular filtered manifold (M^o, D) of type \mathfrak{m} admits a Cartan connection on M^o of type (\mathfrak{g}, H) .

Since M is a uniruled projective manifold and the subvariety $\text{Sing}(D)$ has codimension at least two in M , by Proposition 4.6, there is a rational curve $f: \mathbb{P}^1 \rightarrow M$ such that $f(\mathbb{P}^1) \cap \text{Sing}(D) = \emptyset$ and $f^*T(M)$ is positive. We apply Proposition 4.5 to $M^o = M - \text{Sing}(D)$; thus, the Cartan connection on M^o of type (\mathfrak{g}, H) is locally flat.

To conclude, a geometric structure of M modeled on X is locally equivalent to the standard geometric structure on X . \square

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